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GENERALIZED SOLUTIONS OF THE DYNAMIC PROBLEM OF PERFECT ELASTOPLASTICITY *

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The concept of a generalized solution of an initial boundary value problem for the system of Pradtl-Reuss equations is introduced. It is shown that a generalized solution exists and is unique, and represents within the domain of elasticity a solution of the initial-boundary value problem of the dynamic theory of elasticity. An effective method for the approximate determination of the generalized solution is given, and conditions at its strong discontinuities are obtained. The basic results of this paper were published earlier without proof in /1, 2/.

1. The Prandtl-Reuss equations. Let a perfect elastoplastic body occupy a three-dimensional region Ω with a smooth boundary D. The state of the body is characterized, in Lagrange coordinates, by the stress tensor $\tau_{ij}(t,x)$, the velocity of the body particles $v_i(t,x)$, the elastic strain rate tensor $\varepsilon_{ij}(v) \equiv (v_{i,j}+v_{j,i})/2$ and the plastic strain rate tensor $\lambda_{ij}(t,x)$ ($1 \leqslant i,j \leqslant 3,\ 0 \leqslant t \leqslant T, x \in \Omega$ everywhere). We assume that the measurable part D_1 of the boundary D is free and, that the displacement rate is specified on the part $D_2 = D \setminus D_1$. The density of the body is assumed constant. Assuming that it is equal to unity, we write the equations of elastoplastic flow and initial-boundary conditions /3/ thus

$$a_{ijkh}\tau_{kh} - \varepsilon_{ij}(v) + \lambda_{ij} = 0 \tag{1.1}$$

$$v_i' - \tau_{ij,j} = F_i(t,x) \tag{1.2}$$

$$(\tau_{ij}n_j)(t,x) = 0, \quad x \in D_1; \quad v_i(t,x) = v_i^\circ(t,x), \quad x \in D_2 \tag{1.3}$$

$$\tau_{ij}(0,x) = \tau_{0ij}(x), \quad v_i(0,x) = v_{0i}(x) \tag{1.4}$$

where a_{ijkh} are the coefficients of elasticity, $n_i(x), x \in D$ is the outer normal to Ω, and a prime denotes a time differential. We will supplement (1.1)—(1.4) with the von Mises condition of plasticity /3/ (τ_{ij}^D is the deviator of the tensor τ_{ij})

$$\tau_{ij}^D(t,x)\,\tau_{ij}^D(t,x) \leqslant c_*^2 \tag{1.5}$$

The equations (1.1)—(1.5) are closed by the Prandtl-Reuss relations connecting the stresses with the plastic strain rate

$$\lambda_{ij}(t,x) = \varkappa\sigma_{ij}^D(t,x), \quad \varkappa \geqslant 0$$

where $\varkappa = 0$ when inequality (1.5) is rigorously satisfied. The Prandtl-Reuss relations can be conveniently replaced by the equivalent Drucker postulate /4/. We shall write it in the integrated form

$$\int \lambda_{ij}(t,x)(\tau_{ij}(t,x) - \sigma_{ij}(t,x))\,dx \geqslant 0 \tag{1.6}$$

where σ_{ij} is a tensor field continuously differentiable in $[0,T] \times (\Omega \cup D)$, such that

$$(\sigma_{ij}^D\sigma_{ij}^D)(t,x) \leqslant c_*^2; \quad \sigma_{ij}(t,x)n_j(x) = 0, \quad \forall x \in D_1 \tag{1.7}$$

The initial-boundary value problem (1.1) — (1.7) was studied earlier by Duvaut and Lions, who showed in /6/ the unique solvability of the evolutionary variational inequality following from (1.1) — (1.7), satisfied by the stress tensor integrated with respect to time. Below we apply

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to problem (1.1) – (1.7) the method of monotonic semigroups, enabling us to prove the unique solvability of the problem under somewhat weaker constraints than those shown in /6/ (Theorem 1), to obtain new results concerning the qualitative properties of the solutions (Theorems 2, 4 and 5), and to offer an effective method of deriving an approximate solution (Theorem 3).

Let us assume that the coefficients of elasticity have the symmetry and ellipticity properties

$$a_{ijkh}=a_{jikh}=a_{khij},\quad a_{ijkh}\mu_{ij}\mu_{kh}\geqslant\alpha\mu_{ij}\mu_{ij},\quad \alpha>0,\quad \forall\mu\in l \tag{1.8}$$

where l is the space of symmetric 3×3 matrices. We introduce in l the scalar product $(\sigma,\tau)_l=a_{ijkh}\sigma_{ij}\tau_{kh}$.

Let us define the Hilbert spaces and the scalar products in them as

$$S=L_2(\Omega;R^3),\quad (u,v)_S=\int_\Omega u_j(x)v_j(x)\,dx$$
$$H=L_2(\Omega;l),\quad (\sigma,\tau)_H=\int_\Omega(\sigma(x),\tau(x))_l\,dx$$
$$K=\{\sigma\in H\mid \sigma_{ij,j}\in S\},\quad (\sigma,\tau)_K=(\sigma,\tau)_H+(\sigma_{ij,j},\sigma_{il,l})_S$$

Finally, let K^c be the closure on the norm of the space K of the set of smooth tensor fields $\tau_{ij}(x)$ such, that $\tau_{ij}(x)n_j(x)=0$ when $x\in D_1$.

The natural inclusion $K^c\subset H$ enables us to consider each element $\tau\in H$ as a functional on K^c acting according to the formula $\tau(w)=(\tau,w)_H$, $\forall w\in K^c$.

Let us denote by K^* the component H on the norm of the space conjugate with K^c. Then $K^c\subset H\subset K^*$ and K^* is a subspace of the space of generalized tensor fields on Ω. Let us denote the norms in the space S,H,K and K^* by $|\cdot|_S, |\cdot|_H, |\cdot|_K$ and $|\cdot|_{K^*}$ respectively, and the application of the functional $\eta\in K^*$ to the element $\omega\in K^c$, by $\langle\eta,\omega\rangle$.

Let us find the closed convex subset W of the space K^c

$$W=\{\sigma\in K^c\mid (\sigma_{ij}^D\sigma_{ij}^D)(x)\leqslant c_*^2\}$$

where the inequality holds for almost all $x\in\Omega$. Let us examine the subdifferential of its characteristic function ∂I_W representing a multivalued mapping from W into K^* (see /7/)

$$\partial I_W(\tau)=\{\xi\in K^*\mid\langle\xi,\tau-\mu\rangle\geqslant 0,\ \forall\mu\in W\} \tag{1.9}$$

By virtue of conditions (1.8) we can invert the mapping

$$l\to l,\quad \tau_{ij}\mapsto\sigma_{kh}=a_{khij}\tau_{ij}$$

thus $\tau_{ij}=a^{ijkh}\sigma_{kh}$.

Let us carry out the substitution $v_i=v_i^c-u_i$ and scalar multiply Eq.(1.1) in H by $a^{ijlm}\mu_{lm}(x)$, $\mu\in K^c$. Since

$$(\mu_{lm}n_m)(x)=0,\ x\in D_1;\ u_i(x)=0,\ x\in D_2$$

we have, by virtue of Gauss' formula,

$$(\tau',\mu)_H-(u,\mu_{ij,j})_S+\langle\xi,\mu\rangle=(h,\mu)_H,\quad\forall\mu\in K^c \tag{1.10}$$
$$\xi=a^{ijkh}\lambda_{kh},\quad h_{ij}=a^{ijkh}\varepsilon_{kh}(v^c)$$

From (1.6), (1.9) we obtain the inclusion

$$\xi(t,\cdot)\in\partial I_W(\tau(t,\cdot)) \tag{1.11}$$

Let us now rewrite Eq.(1.2) and initial conditions (1.4), taking into account the substitution made

$$u_i'-\tau_{ij,j}=g_i\equiv F_i-v_i^{c\prime} \tag{1.12}$$

$$\tau_{ij}(0,x)=\tau_{0ij}(x),\quad u_i(0,x)=u_{0i}(x)\equiv v_{0i}(x)-v_i^c(0,x) \tag{1.13}$$

Definition. We shall call the generalized solution of the problem (1.1)–(1.7) the triplet of tensor fields (v_i,τ_{ij},ξ_{ij}), $v_i\equiv v_i^c+u_i$ such, that

$$\tau\in L_\infty(0,T;K^c),\ \tau'\in L_\infty(0,T;H)$$
$$u,u'\in L_\infty(0,T;S),\ \xi\in L_\infty(0,T;K^*) \tag{1.14}$$

for which the inclusions $\tau(t,\cdot)\in W$, (1.11) and Eq.(1.12) hold at almost all $t\in[0,T]$, and initial conditions (1.12) when $t=0$.

Theorem 1. If

$$g,g'\in L_1(0,T;S);\ h,h'\in L_1(0,T;H),\ \tau_0\in W,\ u_0\in S \tag{1.15}$$

and the tensor $u_{0i}(x)$ is such that

$$(u_0, \mu_{ij,j})_S + \langle \eta, \mu \rangle = (\nu, \mu)_H, \quad \forall \mu \in K^\circ \tag{1.16}$$

for some $\eta \in \partial I_W(\tau_0)$, $\nu \in H$, then problem (1.1)–(1.7) has a unique generalized solution. At the same time, if h, g, τ_0, u_0 are replaced, respectively, by $\Delta h, \Delta g, \Delta\tau, \Delta u_0$, so that the conditions (1.15), (1.16) remain valid and the variation in the components of the generalized solution is denoted by $\Delta u_i, \Delta\tau_{ij}$, then the following estimate will hold for any $0 \leqslant t \leqslant T$:

$$(|\Delta u(t,\cdot)|_S^2 + |\Delta\tau(t,\cdot)|_H^2)^{1/2} \leqslant (|\Delta u_0|_S^2 + |\Delta\tau_0|_H^2)^{1/2} + \int_0^t (|h(\tau,\cdot)|_H^2 + |g(\tau,\cdot)|_S^2)^{1/2}\, d\tau$$

Note. Condition (1.16) holds, for example, if $u_{0i,j} \in S$ for all j and $u_{0i}(x) = 0$ for $x \in D_2$. Here we can put $\eta = 0$, $\nu_{ij} = -a_{ijkh}\varepsilon_{kh}(u_0)$.

Definition. The cylinder $Q_0 = (T_1, T_2) \times \Omega_0$, where $0 \leqslant T_1 < T_2 \leqslant T$ and Ω_0 is a smooth subregion of Ω, is called the domain of elasticity of the generalized solution provided that

$$(\tau_{ij}^D \tau_{ij}^D)(t,x) \leqslant c_*^2 - \delta, \quad \delta > 0$$

almost everywhere in Q_0.

Theorem 2. The generalized solution of problem (1.1)–(1.7) constructed in Theorem 1 is the solution of the boundary value problem of dynamic elasticity in the domain of elasticity Q_0, i.e.

$$v_{i,j}(t,x) \in L_2(Q_0), \ \forall i, j \tag{1.17}$$

the equations

$$a_{ijkh}\tau_{kh}' - \varepsilon_{ij}(v) = 0, \quad v_i' - \tau_{ij,j} = F_i \tag{1.18}$$

hold almost everywhere in Q_0, and the boundary conditions

$$(\tau_{ij}n_j)(t,x) = 0, \ x \in \operatorname{int}(D_0 \cap D_1) \tag{1.19}$$

$$v_i(t,x) = v_i^\circ(t,x), \ x \in \operatorname{int}(D_0 \cap D_2) \tag{1.20}$$

are satisfied almost everywhere on $D_0 = D \cap \partial\Omega_0$. Here $\operatorname{int}(D_0 \cap D_i)$ is the interior of the set $D_0 \cap D_i$ in $D, i = 1, 2$.

2. Proof of Theorem 1 and 2.

Consider the mapping

$$\mathrm{DIV}: K^\circ \to S, \ \sigma_{ij} \mapsto \sigma_{ij,j}$$

Let $\mathrm{DIV}^*: S \to K^*$ be its conjugate. We rewrite (1.10) as an equation in terms of K^*

$$\tau' + \mathrm{DIV}^* u + \xi = h \tag{2.1}$$

We consider the multivalued mapping

$$B_0: W \times S \to K^* \times S, (\tau, u) \mapsto (\mathrm{DIV}^* u + \partial I_W(\tau), -\mathrm{DIV}\tau)$$

Let us now define the Hilbert space $L = H \times S$ with scalar product $((\sigma_1, v_1), (\sigma_2, v_2))_L = (\sigma_1, \sigma_2)_H + (v_1, v_2)_S$ and norm $|\cdot|_L$. We consider in L a multivalued mapping B with the domain of definition $D(B)$

$$D(B) = \{(\tau, u) \in W \times S \mid B_0(\tau, u) \cap L \neq \varnothing\}$$

$$B(\tau, u) = B_0(\tau, u) \cap L, \ \forall (\tau, u) \in D(B)$$

Let us denote by $\zeta(t)$ the pair $(\tau(t,\cdot), u(t,\cdot))$ and rewrite (1.11)–(1.13), (2.1) in the form of an equation for $\zeta(t)$ with multivalued non-linearity of B

$$\zeta' + B(\zeta) \ni \varphi(t), \ \zeta(0) = \zeta_0 \tag{2.2}$$

where $\varphi(t) = (h_{ij}(t,\cdot), g_i(t,\cdot))$, $\zeta_0 = (\tau_0, u_0)$. By virtue of the conditions of Theorem 1 we have

$$\varphi, \varphi' \in L_1(0, T; L), \ \zeta_0 \in D(B) \tag{2.3}$$

Definition /8/. The multivalued mapping $E: D(E) \subset L \to L$ will be called monotonic, if

$$(\eta_1 - \eta_2, \theta_1 - \theta_2)_L \geqslant 0, \ \forall \eta_j \in E(\theta_j), \theta_j \in D(E), j = 1, 2$$

and will be maximally monotonic if also

$$(I + \rho E) D(E) = L, \ \forall \rho > 0$$

(here I denotes the unit mapping in L).

Lemma 1. The mapping $B: D(B) \to L$ is maximally monotonic.

Proof. When $(\eta_j, w_j) \in B(\tau_j, u_j)$, $j = 1,2$, the following relation holds:

$$((\eta_1, w_1) - (\eta_2, w_2), (\tau_1, u_1) - (\tau_2, u_2))_L = \langle \mathrm{DIV}^*(u_1 - u_2), \tau_1 - \tau_2\rangle - (\mathrm{DIV}(\tau_1 - \tau_2), u_1 - u_2)_S + \langle \eta_1 - \eta_2, \tau_1 - \tau_2\rangle$$

The sum of the first two terms on the right-hand side of the equation is zero. The third term is non-negative, since $\eta_i \in \partial I_W(\tau_i)$, $i = 1,2$ and the subdifferential ∂I_W is a monotonic mapping /9/.

It remains to confirm the solvability of the following equation for the arbitrary $\rho > 0$, $(\tau, v) \in L$:

$$(\sigma, u) + \rho B(\sigma, u) \ni (\tau, v) \tag{2.4}$$

To do this we consider the following functional on K°:

$$J_\rho(\sigma) = J_\rho^\circ(\sigma) + I_W(\sigma), \quad J_\rho^\circ(\sigma) = |\sigma|_H^2/(2\rho) - (\sigma, \tau)_H/\rho + \int (\rho\sigma_{ij,j}\sigma_{il,l}/2 - \sigma_{ij,j}v_i)\, dx$$

where $I_W(\sigma)$ is a characteristic function of the set $W \subset K^\circ$, equal to zero when $\sigma \in W$ and to $+\infty$ otherwise. The functional J_ρ is convex, semicontinuous from below in the weak topology K°, and coercive. Therefore it has a unique weak minimum $\sigma_0 \in W$ on K°, and the Euler equation

$$\partial J_\rho(\sigma_0) \ni 0$$

holds at the point σ_0 for its subdifferential $\partial J_\rho(\sigma_0)$. But $\partial J_\rho(\sigma) = \partial J_\rho^\circ(\sigma) + \partial I_W(\sigma)$ /7/, therefore we have

$$\sigma_0/\rho + \rho\mathrm{DIV}^*\mathrm{DIV}\sigma_0 + \partial I_W(\sigma_0) \ni \tau/\rho + \mathrm{DIV}^* v \tag{2.5}$$

From (2.5) there follows the inclusion

$$\sigma_0 + \rho\mathrm{DIV}^* u_0 + \rho\partial I_W(\sigma_0) \ni \tau \quad (u_0 = \rho\mathrm{DIV}\sigma_0 + v)$$

and this implies that $(\sigma_0, u_0) \in D(B)$ and inclusion (2.4) holds for (σ_0, u_0).

The lemma proved above, conditions (2.3) and the theory of monotonic semigroups (see /8/, assumption (3.2), (3.3) together imply that problem (2.2) has a unique solution $\zeta(t)$ such that

$$\zeta, \zeta' \in L_\infty(0, T; L)$$

and $\zeta(t) \in D(B)$ for almost all t. From this follows the first assertion of Theorem 1.

To prove the second assertion, we assume that $\zeta^i(t)$ is a solution of (2.2) for $\varphi = \varphi^i(t)$, $\zeta_0 = \zeta_0^i$, $i = 1, 2$. Let us subtract from the equation for $\zeta^1(t)$ the equation for $\zeta^2(t)$, and scalar multiply by $\Delta\zeta(t) = \zeta^1(t) - \zeta^2(t)$ in L. The resulting equation and the monotonic character of B together imply that

$$\tfrac{1}{2}\, d/dt\, |\Delta\zeta(t)|_L^2 \leqslant |\varphi^1(t) - \varphi^2(t)|_L\, |\Delta\zeta(t)|_L$$

Dividing both sides of the equation by $|\Delta\zeta(t)|_L$ and integrating over dt, we obtain the second assertion of the theorem.

Let us now prove Theorem 2. Let $\mu_{ij}(x)$ be a tensor field smooth in $\Omega \cup D$, belonging to the space K° and equal to zero everywhere outside Ω_0. Then $\tau(t, \cdot) \pm \varepsilon\mu \in W$ provided that $\varepsilon > 0$ is sufficiently small for almost all t, and we have, by virtue of (1.9)—(1.11),

$$(\tau', \pm\mu)_H + (u, \pm\mu_{ij,j})_S - (f, \pm\mu)_H \geqslant 0$$

i.e.

$$(u, \mu_{ij,j})_S = (f - \tau', \mu)_H \tag{2.6}$$

Since we can take any smooth symmetric tensor field finite in Ω_0 as $\mu_{ij}(x)$, we have, for almost all t,

$$-\frac{1}{2}(u_{i,j} + u_{j,i})|_{\Omega_0} = a_{ijkh}(f_{kh} - \tau'_{kh})|_{\Omega_0} \tag{2.7}$$

Here the differentiation and contraction on Ω_0 are interpreted as in the theory of generalized functions. From (2.7) and the Korn inequality /5, 6/ it follows that the function $u_{i,j}$ is square integrable on Q_0 for all i, j, and (1.17) follows from this.

Using the smoothness of the tensor $u_i(t, x)$ proved above, we shall apply Gauss' formula to the left side of (2.6). By virtue of (2.7) we obtain

$$\int_{D_2 \cap D_0} \mu_{ij} n_j u_i\, d\gamma = 0$$

where $d\gamma$ is the differential of the element of the surface D. Since the trace $(\mu_{ij}n_j)(x)$, $x \in D_2 \cap D_0$ can be made equal to any tensor from $C_0^\infty(D_2 \cap D_0; R^3)$, therefore $u_i(x) = 0$ when

$x \in \operatorname{int}(D_2 \cap D_0)$ and this yields (1.20). It remains to note that the second equation of (1.18) follows from (1.12), and the boundary condition (1.19) from the fact that the tensor $\tau_{ij}(t, \cdot)$ belongs to the space K°.

3. Approximate construction of the generalized solution. Reducing problem (1.1) — (1.7) to Eq. (2.2) with the maximal monotonic operator B, enables us to use the Trotter formula (/8/, corollary 4.4) for an approximate construction of the generalized solution. We shall use the notation of the proof of Theorem 1, and write

$$\zeta^n(t) = \left(I + \frac{t}{n}B - \frac{t}{n}\varphi\left(\frac{t}{n}(n-1)\right)\right)^{-1} \circ \ldots$$

$$\ldots \circ \left(I + \frac{t}{n}B - \frac{t}{n}\varphi\left(\frac{t}{n}0\right)\right)^{-1}\zeta$$

$$(I + \rho B - \rho\varphi(\theta))^{-1}\omega_* = (\sigma, u)$$

where (σ, u) is the solution of (2.4) for $(\tau, v) = \omega_* + \rho\varphi(\theta)$. The proof of Lemma 1 implies that the determination of $\zeta^n(t)$ is reduced to n variational problems of minimizing the functionals $J_{t/n}(\sigma)$ at different $v_i(x)$, solved one after the other. We find that when $n \gg 1$, the function $\zeta^n(t)$ is a good approximation to the solution of (2.2).

Theorem 3. Let the function $\varphi(t)$ be such, that $\varphi' \in L_\infty(0, T; L)$ and $\zeta_0 \in D(B)$. Then $|\zeta(t) - \zeta^n(t)|_L \leqslant c^{(n)}$ where:

a) if $\varphi(t) \equiv \varphi$ does not depend on t, then

$$c^{(n)} = 2tn^{-1/4}\inf\{|\theta|_L \mid \theta \in \varphi - B(\zeta_0)\}$$

b) in the general case $c^{(n)} = c^{(0)} n^{-1/4}$ where $c^{(0)}$ does not depend on n.

Proof. Assertion a) follows from the maximal monotonicity of the mapping B and corollary 4.4 of /8/. Assertion b) can be obtained from a) by replacing $\varphi(t)$ by a piecewise constant function and using the estimates of the changes in the solution accompanying the changes on the right-hand side of $\varphi(t)$ (see e.g. Lemma 3.1 of /8/).

Assertions analogous to Theorem 3 were proved earlier by P.P. Mosolov in /5/ for the problems of viscoelastoplasticity.

4. Strong discontinuities in the generalized solutions. By analogy with gas dynamics /10/ we shall give the following definition of a strong discontinuity.

Definition. If a hypersurface Γ with edge $\partial\Gamma$ exists in the cylinder $Q_T = (0, T) \times \Omega$ on which the components of the generalized solution of problem (1.1)—(1.7) v, τ have first-order discontinuity, and outside which they are continuous and bounded together with all their derivatives appearing in (1.1), (1.2), then Γ will be called a hypersurface of strong discontinuity and $\gamma(t) = \{x \in \Omega \mid (t, x) \in \Gamma\}$ a surface of strong discontinuity.

If N is a normal to Γ, then the jumps in the tensors v_i, τ_{ij} on Γ in the direction N, will be denoted by $[v]_i$, $[\tau]_{ij}$. If the hypersurface of strong discontinuity Γ is such that $[v] \neq 0$ on Γ (but $[v]$ may possibly vanish on $\partial\Gamma$), then we write $\Gamma = \Gamma^v$.

From now on we will assume everywhere that the tensor $v_i(t, x)$ is continuously differentiable. Then the strong discontinuities (and the strong discontinuity surfaces) of the tensors $v_i(t, x)$ and $u_i(t, x)$ will be equal.

Theorem 4. If $(t_0, x_0) \in \Gamma^v$, then a cylindrical neighbourhood $\omega = (t_1, t_2) \times \omega^x$ of this point on Q_T can be found such, that $\Gamma^v \cap \omega = (t_1, t_2) \times (\gamma^v(t_0) \cap \omega^x)$.

Thus the surface $\gamma^v(t)$ may vary with time only because $\partial\gamma^v(t) \neq \text{const}$ (the discontinuity surface "spreads" in some directions and coalesces in others).

Proof. Let us assume the opposite. Then the projection of the set Γ^v on Ω, which we shall denote by Γ_x^v, has a non-zero Lebesgue measure. The function $t \mapsto v(t, x_0)$ has a discontinuity at the point t_0 for almost all $x_0 \in \Gamma_x^v$, such that $(t_0, x_0) \in \Gamma^v$. Therefore the derivative $v'(t, x)$, regarded as a generalized function, has a non-zero singular component equal to the δ-function with the carrier on Γ^v. This contradicts the smoothness of the function $v(t, x)$ postulated in Theorem 1.

Let $\omega, \Gamma^v, \gamma^v = \gamma^v(t_0)$ be the same as in Theorem 4, ν_j is the normal to γ^v and $C^{-1}(\gamma^v \cap \omega^x; R^3)$ is a space of continuous linear mappings from $C_0^1(\gamma^v \cap \omega^x)$ onto R^3. We note that by virtue of the theorem 4 $N(t, x) = (0, \nu(x))$.

Lemma 2. We define the continuous operator of taking the trace

$$K \to C^{-1}(\gamma^v \cap \omega^x; R^3), \quad \tau \mapsto \tau_{ij}\nu_j|_{\gamma^v \cap \omega^x}$$

If $\psi_i \in C_0^1(\gamma_v \cap \omega^x; R^3)$, then $a^{ijkh}(\psi_k\nu_h + \psi_h\nu_k)\delta_{\gamma^v}/2 \in K^*$ where $\delta_{\gamma^v} - \delta$ is a function of the surface γ^v. The assertion of the lemma follows from Gauss' formula (for a greater detail see /11/, Theorem 1.2).

Let $\varepsilon_{ij}^\circ(v) = (v_{i,j} + v_{j,i})/2$ where the derivatives outside Γ are understood in the point-to-point sense, and are continued on Γ with help of the zero. From Gauss' formula we obtain

Lemma 3. Let $\psi \in C_0^\infty(\omega)$. Then we have

$$\psi\, \mathrm{DIV}^* v = -\psi a^{ijkh}\left[([v]_k \nu_h + [v]_h \nu_k)\delta_{\gamma^v}/2 + \mathring{\varepsilon}_{kh}(v)\right]$$

for all $t \in (t_1, t_2)$. Let $\omega_1 \subset \omega$ be a small subdomain and $f \in C_0^\infty(\omega)$ a function such, that $0 \leqslant f \leqslant 1$ and $f \equiv 1$ on ω_1. By virtue of (2.1), (1.11) and (1.9) where we have put $\mu_{ij} = (1-f)\tau_{ij} + f\sigma_{ij}$, we have

$$\langle \xi_{ij}(t,\cdot), f(t,\cdot)(\tau_{ij}(t,\cdot) - \sigma_{ij}(t,\cdot))\rangle \geqslant 0 \tag{4.1}$$

for any tensor $\sigma_{ij}(t,x)$ such that

$$\sigma \in L_\infty(0, T; K^\circ),\ \sigma(t,\cdot) \in W \tag{4.2}$$

for almost all t. Let us express $\xi_{ij}(t,x)$ in terms of u, τ given by (2.1), and taking into account Lemma 3, substitute into (4.1)

$$\frac{1}{2}\int_{\gamma^v} f(t,x)([v]_i \nu_j + [v]_j \nu_i)(\tau_{ij} - \sigma_{ij})\,d\gamma + \int_\omega w_{ij}(\tau_{ij} - \sigma_{ij})\,dx \geqslant 0$$

where $w \in L_2(Q_T; l)$. From this we have, for almost all t,

$$\int_{\gamma^v} f(t,x)([v]_i \nu_j + [v]_j \nu_i)(\tau_{ij} - \sigma_{ij})\,d\gamma \geqslant 0 \tag{4.3}$$

where the tensor σ_{ij} is the same as in (4.2). In particular, substituting $\sigma_{ij} = \tau_{ij} \pm \delta_{ij}$ we find that

$$[v]_i \nu_i = 0 \tag{4.4}$$

everywhere in $\omega_1 \cap \Gamma^v$. Let $\tau_{ij}^{\pm}(t,x) = \tau_{ij}(t, x \pm 0\cdot\nu(x))$ from $(t,x) \in \Gamma^v$. By virtue of Lemma 2 we have

$$(\tau_{ij}^{+}\nu_i)(t,x) = (\tau_{ij}^{-}\nu_i)(t,x),\quad (t,x) \in \Gamma^v,\quad j = 1,2,3 \tag{4.5}$$

and we can substitute in (4.3) τ_{ij}^{+} as well as τ_{ij}^{-}. Let us fix $(t_0, x_0) \in \Gamma^v \cap \omega_1$ and introduce a system of coordinates with unit vectors

$$n^1 = \nu(x_0),\ n^2 = ([v]/|[v]|)(t_0, x_0),\ n^3 = n^1 \times n^2 \tag{4.6}$$

Then by virtue of (4.4) we have

$$([v]_i \nu_j \tau_{ij}^{+})(t_0, x_0) = (|[v]|\,\tau_{12}^{+D})(t_0, x_0)$$

From (1.5) we have

$$\tau_{12}^{+D}(t_0, x_0) \leqslant (c_*/2)^{1/2} \tag{4.7}$$

If the inequality (4.7) is satisfied strictly, then $\alpha_1 > 0$, the neighbourhoods ω_2, ω_3 of the point (t_0, x_0), $\omega_3 \subset \omega_2 \subset \omega_1$ and the tensor σ_{ij} can all be found such, that

$$(\sigma_{ij}[v]_i \nu_j)(t,x) \geqslant (\sigma_{12}^D |[v]|)(t_0, x_0) - \alpha_1 \geqslant$$
$$(\tau_{12}^{-D}|[v]|)(t_0, x_0) - 2\alpha_1 \geqslant (\tau_{ij}^{+D}[v]_i \nu_j)(t,x) + \alpha_1$$

everywhere in ω_3 and $\sigma = \tau^+$ everywhere in $\omega_1 \setminus \omega_2$. At the same time, when α_1, ω_2 are fixed, a tensor σ and the neighbourhood ω_3 can be chosen such that the quantity $\operatorname{mes}(\omega_2 \setminus \omega_3) = \alpha_2$ will become arbitrarily small. But if the quantity α_2 is sufficiently small, then the left-hand side of inequality (4.3) will be strictly negative. The resulting contradiction shows that $\tau_{12}^{+D}(t_0, x_0) = (c_*/2)^{1/2}$. This together with (1.5), yields the values of all components of the tensor $\tau_{ij}^{-D}(t_0, x_0)$ in the system of coordinates (4.6)

$$\tau_{12}^{+D} = \tau_{21}^{+D} = (c_*/2)^{1/2},\quad \tau_{ij}^{+D} = 0 \ \text{ if } \ \{i,j\} \neq \{1,2\}$$

Therefore, we have, in any orthonormed system of coordinates

$$\tau_{ij}^{+D}(t_0, x_0) = \left(\frac{c_*}{2}\right)^{1/2}\left(\nu_i \frac{[v]_j}{|[v]|} + \nu_j \frac{[v]_i}{|[v]|}\right)(t_0, x_0) \tag{4.8}$$

Repeating the above reasoning for the tensor τ^{-D}, we find that it is also defined by Eq. (4.8). Since the point $(t_0, x_0) \in \Gamma^v$ is arbitrary, this implies the continuity of the deviator of the tensor τ on Γ^v, and (4.5) then implies also the continuity of the whole tensor τ. This proves the following theorem.

Theorem 5. The velocity jump $[v]$ touches the surface of discontinuity γ^v. The stress tensor τ_{ij} is continuous everywhere on Γ^v. When $(t_0, x_0) \in \Gamma^v$, the deviator of the tensor τ is given by formula (4.5).

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THE BOUNDARY LAYER IN THE FLOW OF A PLASTIC MEDIUM NEAR A ROUGH SURFACE*

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High-speed flow of an incompressible plastic medium past a rigid rough surface with slippage along it is investigated. It is assumed that the ratio of the yield point of the medium to the dynamic pressure in the flow is small. An asymptotic representation of the solution is constructed, based on the assumption due to Lavrent'ev that in the case of flows with such properties the principal parts of the velocity and stress fields are represented by the corresponding fields of an ideal fluid. Equations are obtained describing the flow in the boundary layer. Group-theoretic analysis is used to find their solution for flows past wedges and cones. The thickness of the boundary layer is estimated.

1. Let us consider the high-speed flow of an incompressible plastic medium past a fixed impermeable surface, with the particles slipping along the surface. The stresses in the medium satisfy the Mises plasticity condition with constant k /1/. We assume that

$$f = \sqrt{k/(\rho c^2)} \ll 1 \qquad (1.1)$$

where ρ is the density of the medium and c is the characteristic velocity of the flow. Condition (1.1) means that the level of the stress deviator is small compared with the dynamic pressure of the flow. The condition can be satisfied in the flows possessing high deformation rates. It can be expected, by virtue of (1.1), that the velocity of stress fields will differ little from the corresponding fields in the analogous problem for a perfect fluid.

The perfect fluid model was widely used in /2/ in calculating the rigid, intensely deformed materials. In some cases, however, it is useful to know the magnitude of the correction related to the density of the medium. The problem was studied earlier in /3/ for several specific cases. In /4/ expansions of the velocity and stress fields over short distances from the boundary were constructed for the slow flows ($f = \infty$). The boundary layer in a viscoplastic medium was studied in /5—7/ assuming that no slippage of the particles along the boundary took place.

Below, using the results of /3/, we obtain equations describing the flow in the boundary layer, differing appreciably from the corresponding equations for viscous flow and /5—7/ and use them as the starting concepts. Group-theoretic analysis methods /8/ are used to obtain

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